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On Products of Completion Regular Measures

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Some special machinery is introduced for studying products of completion regular measures, the most important tool being the (O_2) condition. This condition was suggested by a technique of the author and C. Gryllakis (*Illinois J. Math.* **35** (1991), 260–268) used to prove that, under some set-theoretic restrictions, the product of two compact completion regular measure spaces is completion regular, provided that one of the topological factors is an arbitrary product of metric spaces. © 1992 Academic Press, Inc.

0. INTRODUCTION

0.1. A completion regular measure in a topological space is a Baire measure μ such that every Borel set is μ-measurable. It is known that the product of two completion regular measures is not, always, completion regular [5] and that the product of two completion regular Radon measures, each on a product of separable metric spaces, is completion regular [9].

From the previous results, concerning measures on arbitrary topological spaces (i.e., not necessarily products of metric spaces), it is natural to look for stronger conditions than the completion regularity, which can be preserved on the products.

In this note the (O_2) condition is introduced and discussed, in the setting of τ-additive measure spaces. In Section 1 a couple of examples of measure spaces satisfying this condition are given: a completion regular measure on any product of separable metric spaces as well as a Haar measure on any compact group (Propositions 1.3 and 1.4). These propositions improve the main results of [8, 9], since a measure space satisfying (O_2) possesses the atribute of having completion regular products (with an arbitrary completion regular measure space).

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Section 2 is devoted to the study of the product of measures satisfying (O_2). A preliminary result (Lemma 2.1) states that the class of all measures satisfying the (O_2) condition is closed under (finite) products. As a consequence, the following theorem is deduced: the product of an arbitrary family of supporting measures, each satisfying (O_2), satisfies again (O_2). This result contains a Choksi and Fremlin theorem [2, Theorem 2] as a special case and also yields a recent theorem of Gryllakis and Koumoullis [10] concerning product measures.

0.2. All measures mentioned in this paper will be nonnegative Borel probability (τ-additive) measures on Hausdorff topological spaces.

Let X be a (completely regular Hausdorff) space. The family of Borel sets in X is the σ-algebra $\mathscr{B}(X)$ generated by the closed sets in X. Let $\{f_\alpha\}$ be a family of real functions on X. We shall say that a subset B of X is determined by $\{f_\alpha\}$, if B belongs to the σ-algebra generated by the sets of the form $f^{-1}(E)$, where E is a Borel set in $\mathbb{R}$.

A Borel measure μ on X is called completion regular (resp. τ-additive), if the completion of the Baire restriction and the completion $\mathscr{B}_\mu(X)$ of μ coincide (resp. for every family $(V_\alpha)_{\alpha\in A}$ of open sets in X,

$$\sup\{\mu(U_{\alpha\in\Gamma} V_\alpha)\colon \Gamma\subset A \text{ countable}\} = \mu(U_{\alpha\in A} V_\alpha)).$$

A Borel τ-additive measure is upper regular with respect to open sets (for information on τ-additive measures we refer to [6; 7, Sect. A7; 15].

A Borel τ-additive probability measure space is a couple (X, μ), where X is a space and μ is a Borel τ-additive probability measure on X. Notice that if p is a continuous mapping from X into a space Y, then the couple $(Y, p(\mu))$ is such a measure space, where $p(\mu)$ is defined by $p(\mu)(A) := \mu(p^{-1}(A))$, $A\varepsilon\mathscr{B}(Y)$.

Let $(X_i)_{i\in I}$ be a family of spaces, $X=\prod_{i\in I} X_i$, and J a subset of I. We say that a subset A of X depends on (the coordinates from) J, if A is of the form $A=pr_J^{-1}(C)$, where pr_J is the canonical projection from X onto $\prod_{i\in J} X_i$ and C is a subset of $\prod_{i\in J} X_i$ (in this case, we usually say that A factors through $\prod_{i\in J} X_i$). In addition, X depends on $\varnothing$. We also say that an open set V in X is basic open, if it is of the form $V=pr_F^{-1}(\prod_{i\in F} V_i)$, where F is a finite subset of I and each V_i (basic) open in X_i.

1. The (O_2) Condition

1.1. Let (Y, ν) be a Borel τ-additive probability measure space. Consider the following conditions:

(O_1) for every open set U in Y there is an open Baire set U' containing U with $U \tilde{_\nu} U'$ (where $\tilde{\nu}$ denotes equivalence modulo ν-negligible sets),

(O_2) for every Borel τ-additive probability measure space (X, μ) and every open set U in $X \times Y$ there is a couple (Z, π), where Z is a separable metric space and π a continuous open surjection from Y onto Z such that $U_{\widetilde{\mu \otimes \nu}}\, p^{-1}p(U)$, where p denotes the mapping $X \times Y \to X \times Z$: $(x, y) \to (x, \pi(y))$ and $\mu \otimes \nu$ the τ-additive product measure of μ and ν.

Remarks and Examples 1.2. The (O_2) property was suggested by a technique used in [8] to verify the completion regularity of a product measure, where the one factor is a product of compact metric spaces.

Assume the notations in 1.1.

(i) Clearly, every measure having the (O_2) property satisfies (O_1) and is completion regular. Also, if Y is a separable metric space, then (Y, ν) satisfies (O_1).

(ii) If (Y, ν) satisfies (O_2) and (X, μ) is completion regular, then the τ- additive product measure $\lambda = \mu \otimes \nu$ is completion regular. Let U, Z, p be as in (O_2). Because of the metrizability of Z, the measure $\kappa = \mu \otimes \pi(\nu)$ is completion regular, so, there is a Baire set E in $X \times Z$ containing $p(U)$ such that $\kappa(p(U)) = \kappa(E)$. We easily verify that $p^{-1}(E)$ is a Baire cover of U (in $X \times Y$) with $\lambda(p^{-1}(E)) = \lambda(U)$. Since U is an arbitrary open set in $X \times Y$, the completion regularity of λ follows.

(iii) Suppose that Y is a product of separable metric space and ν completion regular. Then ν satisfies (O_1). This has been established for compact measure spaces in [2, Lemma 2, p. 121]. However, the arguments of the proof there apply unchanged for arbitrary τ-additive measure spaces.

(iv) If Y is a compact topological group and ν the Haar measure on Y, then ν satisfies (O_1)—see, e.g., [3, p. 84].

The following two propositions show that the class of measures satisfying (O_2) is much larger than that of measures on separable metric spaces. Notice that the proof of the first does not invoke the key lemma in [9, Sect. 3, p. 565].

PROPOSITION 1.3. *Every Borel τ-additive completion regular probability measure on (a space homeomorphic to) a product of separable metric spaces satisfies* (O_2).

Proof. Consider a family $(Y_j)_{j \in J}$ of separable metric spaces. For $j \in J$ fix a countable base $\mathscr{B}_j$ for the topology of Y_j and for $R \subset J$ write Ω_R for the set of sets W of the form $W = \bigcap_{i \in I} pr_i^{-1}(B_i)$, where $I \subset R$ is finite, $B_i \in \mathscr{B}_i$ for $i \in I$, and pr_i is the projection from $X \times \prod_{i \in J} Y_i$ onto Y_i. Let ν be a

τ-additive completion regular probability measure on $Y=\prod_{j\in J} Y_j$, (X, μ) an arbitrary τ-additive probability measure space, and A a subset of $X\times Y$ measurable with respect to $\lambda=\mu\otimes\nu$ with $\lambda(A)>0$. Set $L_A := \{F$ open in $X\times Y$: F factors through X, $\lambda(F\cap A)>0\}$.

Claim. There exists a countable subset R_A of J such that $\lambda(F\cap A\cap W\cap V)>0$ for all open sets W depending on R_A with $\lambda(A\cap W)>0$, all $F\in L_{A\cap W}$, and all nonempty open sets V that depend on $J-R_A$.

Without loss of generality, we can assume that A is λ-supporting (i.e., $\lambda(A\cap H)>0$ whenever H is an open set meeting A). Suppose, if possible, that the claim is not true. Then, for every countable subset R of J there exist $W_R\in\Omega_R$, $V_R\in\Omega_{J-R}$, and $F_R\in L_{A\cap W_R}$ satisfying the condition

$$\lambda(F_R\cap A\cap W_R)>0 \qquad \text{and} \qquad \lambda(F_R\cap A\cap W_R\cap V_R)=0. \qquad (\mathrm{I}_1)$$

Therefore, by induction on ω_1 (the first uncountable ordinal), starting with an arbitrary $R_0\subset J$ countable, we find a strictly increasing family $\{R_\alpha, \alpha<\omega_1\}$ of countable subsets of J such that

$$\lambda(F_{R_\alpha}\cap A\cap W_{R_\alpha})>0 \qquad \text{and} \qquad \lambda(F_{R_\alpha}\cap A\cap W_{R_\alpha}\cap V_{R_\alpha})=0, \quad \alpha<\omega_1. \qquad (I_2)$$

By the Δ-system lemma (see [12, Lemma 22.6] or [4, p. 5]), we can find an uncountable subset E of ω_1 and $I_0\subset J$ finite such that $I_{R_\alpha}\cap I_{R_\beta}=I_0$, $\alpha, \beta\in E$. Because the B_j's are countable, there exist $W_0\in\Omega_{I_0}$ and an uncountable subset E' of E such that

$$W_{R_\alpha}=W_0\cap W'_\alpha, \qquad \alpha\varepsilon E' \qquad (\mathrm{I}_3)$$

(where the W'_α's are open sets that depend on $J-I_0$). Then, by (I_2) and (I_3),

$$\lambda(F_{R_\alpha}\cap A\cap W_0\cap W'_\alpha)>0 \qquad \text{and} \qquad \lambda(F_{R_\alpha}\cap A\cap W_0\cap W'_\alpha\cap V_{R_\alpha})=0,$$
$$\text{for all} \quad \alpha\in E'. \qquad (\mathrm{I}_4)$$

Consider now any $\Gamma\subset E'$ uncountable and set $D:=A\cap W_0\cap(U_{\alpha\in\Gamma}F_{R_\alpha})$. Then, clearly,

$$\lambda(F_{R_\alpha}\cap D\cap W'_\alpha)>0 \qquad \text{and} \qquad \lambda(F_{R_\alpha}\cap D\cap W'_\alpha\cap V_{R_\alpha})=0,$$
$$\text{for all} \quad \alpha\in\Gamma. \qquad (\mathrm{I}_5)$$

Since A is λ-supporting, for every $\alpha\in\Gamma$,

$$pr_X(F_{R_\alpha})\cap D^u=\varnothing, \qquad \text{for all} \quad u\in pr_Y(W'_\alpha\cap V_{R_\alpha}) \qquad (\mathrm{I}_6)$$

(where pr_X, resp. pr_Y, denotes the canonical projection from $X \times Y$ onto X, resp. Y, and for $u \in Y$, $D^u := \{t \in X : (t, u) \in D\}$).

Let $M = \{u \in Y : \mu(D^u) > 0\}$. By Fubini's theorem for τ-additive measures (see [1, 13]), $\nu(M) > 0$. Since ν is completion regular, there are a nonempty Baire subset N of M and a countable subset of J, on which N depends (see Remark 1.2(iii)). Because the sets $V_\alpha = pr_Y(W'_\alpha \cap V_{R_\alpha})$, $\alpha \in \Gamma$, depend on pairwise disjoint subsets of J and Γ is uncountable, N and V_α depend on pairwise disjoint subsets of J, for all but a countable number of $\alpha \in \Gamma$.

Let $\Gamma' := \{\alpha \in \Gamma : N \text{ and } V_\alpha \text{ depend on disjoint subsets of } J\}$ and choose any point $y \in N \cap (\bigcap_{\alpha \in \Gamma'} V_\alpha)$. Then, by (I_6) we have that $pr_X(F_{R_\alpha}) \cap D^Y = \varnothing$, for $\alpha \in \Gamma'$. Since $\mu(D^Y) > 0$, we deduce

$$0 < \mu(U_{\alpha \in \Gamma'}\, pr_X(F_{R_\alpha})) = \lambda(U_{\alpha \in \Gamma'} F_{R_\alpha})$$
$$< \mu(U_{\alpha \in \Gamma}\, pr_X(F_{R_\alpha})) = \lambda(U_{\alpha \in \Gamma} F_{R_\alpha}).$$

Thus,

> for every uncountable subfamily $\{F_{R_\alpha}, \alpha \in \Gamma\}$ of the family $\{F_{R_\alpha}, \alpha \in E'\}$ there exists a countable subset Δ of Γ such that $0 < \lambda(U_{\alpha \in \Gamma - \Delta} F_{R_\alpha}) < \lambda(U_{\alpha \in \Gamma} F_{R_\alpha})$. $(*)$

But, $(*)$ contradicts the fact that λ is a probability measure. This ends the proof of the claim.

To complete the proof of Proposition 1.3, take an open set U in $X \times Y$ (we can assume that its complement U^c in $X \times Y$ is λ-supporting). There is $R := R_{U^c} \subset J$ countable satisfying the claim (for the set U^c in place of A). Let $x \in U$ and an open neighborhood W of x disjoint from U^c. Express W as $W_1 \cap W_2$, where W_1 factors through $X \times \prod_{k \in R} Y_k$, W_2 depends on $J - R$. By the choice of R, W_1 must be disjoint from U^c. Since x is an arbitrary point of U, U again factors through $X \times \prod_{k \in R} Y_k$.

We set $Z := \prod_{k \in R} Y_k$, $\pi :=$ the canonical projection from Y onto Z, and easily verify that (Z, π) satisfies (O_2). ▌

The proof of the next proposition is in the spirit of [2, Theorem 3]. Our method is to use the fact that a (σ-compact) group is a suitable projective limit of separable metric spaces.

PROPOSITION 1.4. *The Haar measure on an arbitrary compact topological group satisfies* (O_2).

Proof. Consider a compact group Y equipped with a (left) Haar measure ν. Let (X, μ) be a measure space, $\lambda = \mu \otimes \nu$, and U an open set in $X \times Y$. We propose to show that there exists a couple (Z, π) satisfying (O_2).

There is a family of open rectangles $\mathscr{B} = \{W_l \times V_l, i \in L\}$—i.e., each W_l is open in X and each V_l open in Y—with $U = U\{W_l \times V_l : l \in L\}$.

Since ν satisfies (O_1) (see Remark 1.2(vi)), without loss of generality, we can assume that V_l is Baire in Y, for every $l \in L$. On the other hand, because of the τ-additivity of λ, there is a sequence $\{W_n \times V_n\}$ in $\mathscr{B}$ such that

$$\lambda(U) = \lambda(\bigcup_{n \in \mathbb{N}} W_n \times V_n).$$

Set $V := \bigcup_{n \in \mathbb{N}} W_n \times V_n$. Therefore, since $\{V_n\}$ is a countable family of Baire subsets of Y, we can find a sequence of continuous functions on Y which determines all sections V_t, $t \in X$ (where, for a subset G of $X \times Y$ and $t \in X$, the section of G by t is the set $G_t = \{u \in Y : (t, u) \in G\}$).

Now, by arguments used in [3, p. 84], we find a compact normal subgroup Y_0 of Y with the properties:

(a) The quotient space $Z = Y/Y_0$ is metrizable,

(b) each V_t is of the form $V_t = \pi^{-1}(B(t))$, for some Borel set $B(t)$ in Z (where π denotes the canonical projection from Y onto Z).

Let $p; X \times Y \to X \times Z : (x, y) \to (x, \pi(y))$.

We claim that this couple (Z, π) satisfies (O_2). In fact, using again Fubini's theorem (for τ-additive measures), we have

$$\lambda(p^{-1}p(U)) = \int \nu((p^{-1}p(U))_t)\, d\mu(t) \qquad (\text{because } (p^{-1}p(U))_t = \pi^{-1}\pi(U_t))$$

$$= \int \nu(\pi^{-1}\pi(U_t))\, d\mu(t), \tag{i_1}$$

and

$$\lambda(U) = \int \nu(U_t)\, d\mu(t). \tag{i_2}$$

Also, because $\lambda(V) = \lambda(U)$, we have $\int \nu(V_t)\, d\mu(t) = \int \nu(U_t)\, d\mu(t)$. Thus, μ-almost for all $t \in X$, V_t is an open Baire kernel of U_t such that $\nu(V_t) = \nu(U_t)$. So (see the proof in [3, p. 84]),

$$\nu(\pi^{-1}\pi(U_t)) = \nu(U_t) \qquad \text{a.e. } (\mu). \tag{i_3}$$

We deduce $\lambda(p^{-1}p(U)) = \lambda(U)$. ∎

Question 1.5. Let G be an infinite compact totally disconnected topological group and μ a completion regular measure on G. Then (G, μ) satisfies (O_2), since such a G is homeomorphic to a dyadic space $\{0, 1\}^\gamma$, where γ is some cardinal (see [11, (9.15) theorem]). But, we do not know what the situation is for completion regular measures on arbitrary compact groups. Proposition 1.4 does not help in that direction.

Now, by Remark 1.2(ii), we immediately obtain the following corollaries. The first is an earlier result of Gryllakis and Koumoullis [10].[1]

COROLLARY 1.6. *Let (X, μ) be a Borel τ-additive completion regular probability measure space and ν a τ-additive completion regular probability measure on an infinite product of separable metric spaces. Then $\mu \otimes \nu$ is completion regular.*

COROLLARY 1.7. *Let (X, μ) be a Borel τ-additive completion regular probability measure space and ν the Haar measure on a compact group. Then, $\mu \otimes \nu$ is completion regular.*

2. ON PRODUCT MEASURES

The (Radon) product of two completion regular probability Radon measures is not, in general, completion regular [5]. However, the following elementary lemma asserts that the class of measures satisfying (O_2) is productive.

Before stating Lemma 2.1, we need some notations: Let $(p_i)_{i \in I}$ be a family of mappings $P_i: A_i \to B_i$. Then, $\times_{i \in I} p_i$ denotes the mapping $\prod_{i \in I} A_i \to \prod_{i \in I} B_i$: $(a_i)_{i \in I} \to (p_i(b_i))_{i \in I}$. Also, for a set X, I_X is the identity map from X to X.

LEMMA 2.1. *The product of two measures satisfying (O_2) satisfies again (O_2).*

Proof. Let (Y_1, ν_1), (Y_2, ν_2) be (O_2) spaces, $Y = Y_1 \times Y_2$, $\nu = \nu_1 \otimes \nu_2$, (X, μ) a Borel τ-additive probability measure space and U a nonempty open set in $X \times Y$.

There is a couple (Z_1, π_1) such that $U \sim (I_{X \times Y_2} \times \pi_1)^{-1} (I_{X \times Y_2} \times \pi_1)(U)$ (with respect to $\mu \otimes \nu$). On the other hand, there is a couple (Z_2, π_2) such that $(I_{X \times Y_2} \times \pi_1)(U) \sim (I_{X \times Z_1} \times \pi_2)^{-1} (I_{X \times Z_1} \times \pi_2)(I_{X \times Y_2} \times \pi_1)(U)$ (with respect to $\mu \otimes \pi_1(\nu_1) \otimes \nu_2$).

We set $Z := Z_1 \times Z_2$, $\pi := \pi_1 \times \pi_2$, and easily verify that (Z, π) satisfies (O_2) for the measure spaces (X, μ) and (Y, ν). ∎

[Note. According to [6, Theorem 4F], if $(X_i, \mu_i)_{i \in I}$ is a family of Borel probability τ-additive measure spaces, then there is a unique Borel τ-additive measure on $\prod_{i \in I} X_i$ extending the simple product of the μ_i on

[1] This has, also, been already proved, under Martin's Axiom and the negation of the continuum hypothesis, by C. Gryllakis and the author in [8].

the product σ-algebra $\bigotimes_{i\in I} B(X_i)$. In the sequel, this product measure will be denoted by $\bigotimes_{i\in I}\mu_i$.]

Every measure $\bigotimes_{i\in I}\mu_i$ is completion regular, provided that each μ_i is a completion regular measure supported on a space homeomorphic to a product of separable metric spaces [10]. The next theorem implies this recent result.

THEOREM 2.2. *The product of an arbitrary family of supporting measures satisfying* (O_2) *satisfies again* (O_2).

Proof. Let (X,μ) be a Borel τ-additive probability measure space, (Y_i, ν_i), $i\in J$, a family of spaces satisfying (O_2) with $\operatorname{supp}\nu_i = Y_i$ for every $i\in J$, $Y=\prod_{i\in J} Y_i$, $\nu=\bigotimes_{i\in J}\nu_i$, $\lambda=\mu\otimes\nu$.

Claim. For every closed λ-supporting set K there exists a countable subset $I=I_K$ of J such that K factors through $X\times\prod_{i\in I} Y_i$.

Let K be a closed λ- supporting set. There are a countable subset I of J and a closed cover L of K such that L factors through $X\times\prod_{i\in I} Y_i$ and $\lambda(K)=\lambda(L)$. Take any $x\in(X\times Y)-K$ and a basic open neighborhood W of x disjoint from K. Express W as $G\cap V$, where G factors through $X\times\prod_{i\in I} Y_i$ and V depends on $J-I$. Then, $\lambda(K\cap W)=\lambda(L\cap W)=(I_X\times pr_I)(\lambda)(L\cap G)\cdot pr_{J-I}(\lambda)(V)=0$ (where for a subset D of J, pr_D denotes the projection from $X\times Y$ onto $\prod_{i\in D} Y_i$). Therefore, $\lambda(K\cap G)=0$, so $K\cap G=\varnothing$. Since x is an arbitrary point on $(X\times Y)-K$, K factors through $X\times\prod_{i\in I} Y_i$.

Consider now an open set U in $X\times Y$. We propose to show that there is a couple (Z,π) as in (O_2). We can assume that $(X\times Y)-U$ is supporting. In this case, by the claim, there is a countable subset Q of J such that $U=(I_X\times pr_Q)^{-1}(I_X\times pr_Q)(U)$. It follows (taking $(I_X\times pr_Q)(U)$ in place of U) that we can assume J to be countable.

Let $J=\{i_1, i_2, \dots\}$. For simplicity, we set $Y_n := Y_{i_n}$, $\nu_n := \nu_{i_n}$, $n\in\mathbb{N}$. Moreover, we set $Z_0 := Y_1$.

There exists a couple (Z_1,π_1) (in place of (Z,π)) satisfying the condition (O_2)—for $(X\times\prod_{k\geqslant 2} Y_k, \mu\otimes\bigotimes_{k\geqslant 2}\nu_k)$ in place of (X,μ) and (Y_1,ν_1) in place of (Y,ν). By induction on $n\in\mathbb{N}$, we construct a sequence of couples $\{(Z_n,\pi_n)\}$ such that

(1) for each $n\in\mathbb{N}$, Z_n is a separable metric space and π_n an open continuous surjection from Y_n onto Z_n,

(2) for each $n\geqslant 2$, (Z_n,π_n) (in place of (Z,π)) satisfies 1.2(ii) (for the open set $(I_{X\times\prod_{k\leqslant n-2} Z_k}\times\pi_{n-1}\times I_{\prod_{k\geqslant n} Y_k})(U)$ in place of U, (Y_n,ν_n) in

place of (Y, ν), and $(X\times\prod_{k\leqslant n-1} Z_k\times\prod_{k\geqslant n+1} Y_k, \mu\otimes\bigotimes_{k\leqslant n-1}\pi_k(\nu_k)\otimes\bigotimes_{k\geqslant n+1}\nu_k)$ in place of (X, μ)).

Now, we set $Z:=\prod_{n\in\mathbb{N}} Z_n$, $\pi:=\times_{n\in\mathbb{N}}\pi_n$. We shall check that (Z, π) is the required couple. For this, it suffices to verify that $(I_X\times\pi)^{-1}(I_X\times\pi)(U)\sim_\lambda U$.

Write $U=\bigcup_{n\in\mathbb{N}} U_n$, where each U_n has the form $U_n=V_n\times\prod_{i\geqslant n+1} Y_i$ for some open set V_n in $X\times\prod_{i\leqslant n} Y_i$. By the construction of the (Z_n, π_n), we have

$$\begin{aligned}(I_X\times\pi)^{-1}(I_X\times\pi)(U)&=\bigcup_{n\in\mathbb{N}}(I_X\times\pi)^{-1}(I_X\times\pi)(U_n)\\&=\bigcup_{n\in\mathbb{N}}\Big(I_X\times\mathop{\times}_{k\leqslant n}\pi_k\times I_{\prod_{k\geqslant n+1}Y_k}\Big)^{-1}\\&\quad\times\Big(I_X\times\mathop{\times}_{k\leqslant n}\pi_k\times I_{\prod_{k\geqslant n+1}Y_k}\Big)(U_n)\\&\subset\bigcup_{n\in\mathbb{N}}\Big(I_X\times\mathop{\times}_{k\leqslant n}\pi_k\times I_{\prod_{k\geqslant n+1}Y_k}\Big)^{-1}\\&\quad\times\Big(I_X\times\mathop{\times}_{k\leqslant n}\pi_k\times I_{\prod_{k\geqslant n+1}Y_k}\Big)(U)\sim_\lambda U.\quad\blacksquare\end{aligned}$$

Note added in proof. In joint work with D. H. Fremlin, we have proved that every completion regular measure on a compact group satisfies (O_2).

REFERENCES

1. A. G. Babiker and J. D. Knowles, Functions and measures on product spaces, *Mathematica* **32** (1985), 60–67.
2. J. R. Choksi and D. H. Fremlin, Completion regular measures on product spaces, *Math. Ann.* **241** (1979), 113–128.
3. J. R. Choksi, Recent developments arising out of Kakutani's work on completion regularity of measures, *in* "Contemp. Math.," Vol. 26, Amer. Math. Soc., Providence, RI, 1984.
4. W. W. Comfort and S. Negrepontis, "Chain Conditions in Topology," Cambridge Univ. Press, London/New York, 1982.
5. D. H. Fremlin, Products of Radon measures: a counter example, *Canad. Math. Bull.* **19** (1976), 285–289.
6. D. H. Fremlin, Note of 2/6/82.
7. D. H. Fremlin, "Consequences of Martin's Axiom," Cambridge Univ. Press, London/New York, 1984.
8. S. Grekas and C. Gryllakis, Completion regular measures on product spaces with application to the existence of Baire strong liftings, *Illinois J. Math.* **35** (1991), 329–344.
9. C. Gryllakis, Products of completion regular measures, *Proc. Amer. Math. Soc.* **103**, No. 2 (1988).

10. C. GRYLLAKIS AND G. KOUMOULLIS, Completion regularity and τ-additivity of measures on product spaces, *Compositio Math.* **73** (1990), 329–344.
11. E. HEWITT AND K. ROSS, "Abstract Armonic Analysis, I," Springer-Verlag, Berlin, 1963.
12. T. JECH, "Set Theory," Academic Press, New York, 1978.
13. J. D. KNOWLES, Measures on topological spaces, *Proc. London Math. Soc.* **17** (1967), 139–156.
14. P. RESSEL, Some continuity and measurability results on spaces of measures, *Math. Scand.* **40** (1977), 69–78.
15. V. S. VARADARAJAN, Measures on topological spaces, *in* "Amer. Soc. Transl.," Vol. 48, pp. 161–228, Amer. Math. Soc., Providence, RI, 1965.